not in either static or dynamic equilibrium.

(b) Lagrange's Equations from D'Alembert's Principle :

The co-ordinate transformation equations are

$$\mathbf{r}_i = \mathbf{r}_i\,(q_1, q_2 \ldots, q_n, t),$$

so that

$$\frac{d\mathbf{r}_i}{dt} = \frac{\partial \mathbf{r}_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial \mathbf{r}_i}{\partial q_2}\frac{dq_2}{dt} + \ldots + \frac{\partial \mathbf{r}_i}{\partial t}\frac{dt}{dt}$$

or

$$\mathbf{v}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t} \qquad \ldots (3)$$

Further infinitesimal displacement $\delta \mathbf{r}_i$ can be connected with δq_j as

$$\delta \mathbf{r}_i = \sum_i \frac{\partial \mathbf{r}_i}{\partial q_j}\delta q_j + \frac{\partial \mathbf{r}_i}{\partial t}\delta t$$

But last term is zero since in virtual displacement only co-ordinate displacement is considered and not that of time. Therefore

$$\delta \mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j}\delta q_j$$

Now we write eq. (2) as $\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = 0,$

$$\sum_{ij} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j - \sum_{ij} \dot{\mathbf{p}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = 0.$$

We define the term

$$\sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = Q_j,$$

the components of the generalised force. As discussed under 'generalised force' art. 1.10, q's n... *not have the dimensions of length, and similarly, it is not necessary for the Q's to have ... dimensions of force, but it is necessary that the product $Q_j \delta q_j$ must have the dimensions of wo...*

Thus above equation takes the form

$$\sum_j Q_j \delta q_j - \sum_{ij} \dot{\mathbf{p}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = 0. \quad \text{...(}$$

Let us evaluate the second term of above eq. (4) :

$$\sum_{ij} \dot{\mathbf{p}} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = \sum_{ij} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j$$

$$= \sum_{ij} \left\{ \frac{d}{dt} \left(m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) - m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) \right\} \delta q_j$$

$$= \sum_{ij} \left\{ \frac{d}{dt} \left(m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) - m_i \mathbf{v}_i \cdot \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) \right\} \delta q_j \quad \text{... (5}$$

Further

$$\frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) = \sum_k \frac{\partial}{\partial q_k} \left(\frac{\partial \mathbf{r_i}}{\partial q_j} \right) \frac{dq_k}{dt} + \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) \frac{dt}{dt}$$

$$= \sum_k \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial q_j} \dot{q}_k + \frac{\partial^2 \mathbf{r}_i}{\partial t \partial q_j}$$

$$= \sum_k \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial q_k} \dot{q}_k + \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial t}$$

$$= \frac{\partial}{\partial q_j} \left(\frac{d\mathbf{r}_i}{dt} \right) = \frac{\partial \mathbf{v_i}}{\partial q_j} \quad \text{... (6)}$$

Also differentiating eq. (3) with respect to $\dot{q}_j$, we get

$$\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j} \quad \text{... (7)}$$

Putting eqs. (6) and (7) in eq. (5), we get

$$\sum_{ij} \dot{\mathbf{p}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = \sum_{ij} \left\{ \frac{d}{dt} \left(m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} \right) - m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j} \right\} \delta q_j$$

$$= \sum_j \left[\frac{d}{dt} \left\{ \frac{\partial}{\partial \dot{q}_j} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right\} - \frac{\partial}{\partial q_j} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right] \delta q_j.$$

With this substitution, eq. (4) becomes

$$\sum_j Q_j \delta q_j - \sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j = 0,$$

or

$$\sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0.$$

where for $\sum_i \frac{1}{2} m_i v_i^2$, T is written since it represents the total kinetic energy of the system.

Since the constraints are holonomic, q_j are independent of each other and hence to satisfy above equation the coefficient of each δq_j should separately vanish, *i.e.*,

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] = Q_j. \quad \ldots (8)$$

As j ranges from 1 to N, there will be N such second order equations.

EC-(3)

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