(a) D'Alembert's principle :

This method is based on the *principle of virtual work*. The system is subjected to an infinitesimal displacement consistent with the forces and constraints imposed on the system at the given instant t. This *change in the configuration of the system is not associated with a change in time, i.e.*, there is no actual displacement during which forces and constraints may change and hence the displacement is termed as virtual displacement.

Now suppose the system is in equilibrium, *i.e.*, total force $\mathbf{F}_i$ on every particle is zero; then work done by this force in a small virtual displacement $\delta \mathbf{r}_i$ will also vanish. *i.e.*, for whole system for N particles

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0.$$

Let this total force be expressed as sum of applied force $\mathbf{F}_i^a$ and forces of constraints $\mathbf{f}_i$. Then above equation takes the form

$$\sum_i \mathbf{F}_i^a \cdot \delta \mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0.$$

We now consider the system for which the virtual work of the forces of constraints is zero. An example of such a system can be that of a particle which is constrained to move on a smooth surface so that the forces of constraints are perpendicular to the surface while virtual displacement is tangential to it. In such a situation virtual work done by forces of constraints will be zero.

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* Let us illustrate the point :

If we solve second order equation $\dfrac{d^2y}{dx^2} = 0$, equation of a straight line is obtained, *viz.*, $y = mx + c$. Obviously it has two constants of integration. If dy/dx is defined, m is defined (like velocity $\dot{q}_j$) and if y is defined c will also be defined (like q_j). Now if m and c both are fixed, straight line will be fixed or specified. For fixing two constants m and c, we require two initial conditions. Suppose only one initial condition is provided, say only value of m is specified, then there is a possibility of various straight lines with same m but different c (intercept on y-axis). Obviously they will be parallel lines. If, instead, value of c is specified but not of m, there is again the possibility of a number of lines with same c but different m (slope). Equation (5), on solving, leads to the equation of path of system and to fix that path three initial values of $\dot{q}_j$ and three initial values of co-ordinates q_j are required; but since configuration space provides only the three initial values of co-ordinates, path will not be fixed but arbitrary.

Thus
$$\sum_i \mathbf{F}_i^a \cdot \delta \mathbf{r}_i = 0. \quad \ldots (1)$$

The equation is termed as principle of virtual work. To interpret the equilibrium of the system, D'Alembert adopted an idea of a reversed force. He conceived that a system will remain in equilibrium under the action of a force equal to the actual force $\mathbf{F}_i$ plus *reversed effective force* $\dot{\mathbf{p}}_i$.

Thus
$$\mathbf{F}_i + (-\dot{\mathbf{p}}_i) = 0$$

or
$$\mathbf{F}_i - \dot{\mathbf{p}}_i = 0,$$

where $-\dot{\mathbf{p}}_i$ appears as an effective force called the *reversed force of inertia* on the *i*th particle, supplementing the already existing externally applied force $\mathbf{F}_i$.

Thus the principle of virtual work takes the form
$$\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0$$

Again writing
$$\mathbf{F}_i = \mathbf{F}_i^a + \mathbf{f}_i,$$
$$\sum_i (\mathbf{F}_i^a - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0.$$

Since forces of constraints are no more in picture, it is better to drop superscript a. Thus
$$\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0, \quad \ldots (2)$$

which is called *D' Alembert's Principle*. The key point is that in eq. (2) we got rid of the forces of constraints. To satisfy eq. (2), we can not equate the coefficients of $\delta \mathbf{r}_i$ to zero since $\delta \mathbf{r}_i$ are not independent of each other and hence it is necessary to transform $\delta \mathbf{r}_i$ changes into the changes of generalised co-ordinates, δq_j, which are independent of each other. The coefficient of every δq_j will then be equated to zero.

Thus we note that, as this principle does not involve forces of constraints in any way, it is sufficient to specify all the applied forces only. Further it is valid for all rheonomic and scleronomic systems that are either holonomic or homogeneous non-holonomic. The inertial force $-\dot{\mathbf{p}}_i$ is introduced to reduce the problem of dynamics to one of statics. This force can be regarded as an inertial force arising in an accelerated frame of reference *e.g.* in a freely falling accelerated frame $\mathbf{F}_i - \dot{\mathbf{p}}_i = 0$, gravity is nullified by the inertial force even though the whole system is in motion and not in either static or dynamic equilibrium.

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