1.6. GENERALISED CO-ORDINATES

To describe the configuration of a system, we select the *smallest possible number of variables*. These are called the *generalised co-ordinates* of the system. We shall not restrict our choice only to cartesian co-ordinates. In many cases these are not the most convenient co-ordinates in terms of which we are to describe the motion of the system. *A set of generalised co-ordinates is any set of co-ordinates which describe the configuration.* We wish, sometimes, to introduce not all co-ordinates with respect to a fixed co-ordinate system but some of them may be selected with respect to a new origin or to a moving co-ordinate system. For example, in dealing with rigid body motion, we specify three cartesian co-ordinates to locate the centre of mass with respect to an external origin and three angle co-ordinates relative to origin at the centre of mass. Thus the generalised co-ordinates should all be chosen the conventional orthogonal position co-ordinates or all may be angle co-ordinates. In fact all sorts of quantities may be impressed to serve as generalised co-ordinates. Thus the amplitudes in a Fourier series expansion a_i may be used as generalised co-ordinates, two angles with the vertical in a double pendulum, the distance s along the path of motion from the equilibrium position in case of a bob of pendulum moving in a vertical plane, or we may find it convenient to employ quantities having dimensions of energy, angular momentum or time.

*Constraints in the case of a rigid body are not independent if we express them in terms of the constancy of mutual distances (see chapter on rigid body motion).

Question arises how to choose a suitable set of generalised co-ordinates in a given situation ? In doing so we must be guided by the following three principles :

(i) *Their values determine the configuration of the system.*

(ii) *They may be varied arbitrarily and independently of each other, without violating the constraints on the system.*

(iii) *There is no uniqueness in the choice of generalised co-ordinates. Then our choice should fall on a set of co-ordinates that will give us a reasonable mathematical simplification of the problem.*

Notation for generalised co-ordinates : Generalised co-ordinates are designated by letter q with numerical subscripts; $q_1, q_2, \dots q_n$ represent a set of n generalised co-ordinates; or, alternatively, by a letter subscript to q and specifying within brackets the numerical values that the letter subscript is allowed to take, *e.g.*, q_j ($j = 1, 2, \dots n$). When we switch over to describe a specific problem, the symbols $q_1, q_2 \dots$ correspond to co-ordinates that we choose to describe the motion. Thus when a particle moves in a plane, it may be described by cartesian co-ordinates x, y or the polar co-ordinates, r, θ and so on, and we write :

$$q_1 = x \qquad \text{or} \qquad q_1 = r = \sqrt{(x^2 + y^2)},$$

$$q_2 = y \qquad\qquad q_2 = \dot{\theta} = \tan^{-1}\frac{y}{x}. \qquad \dots (8)$$

When the problem involves some spherical symmetry, it is suitable to use spherical co-ordinates :

$$q_1 = r = (x^2 + y^2 + z^2)^{1/2},$$

$$q_2 = \theta = \cot^{-1}\frac{z}{(x^2 + y^2)^{1/2}} \qquad \dots (9)$$

$$q_3 = \phi = \tan^{-1}\frac{y}{x}.$$

If it is preferred to accept a co-ordinate system moving uniformly with velocity v in x- direction, generalised co-ordinates are

$$q_1 = x - \dot{x}t.$$

$$q_2 = y \qquad \dot{x} = v = \text{constant}. \qquad \dots (10)$$

$$q_3 = z.$$

For a rod lying on a plane surface, capable of taking any orientation, the suitable choice of co-ordinates to describe the configuration of the rod will be the cartesian co-ordinates ξ and η to locate any point A of the rod and angle θ indicating the orientation of the rod with respect to the co-ordinates axes OX and OY. Then

$$q_1 = \xi,$$

$$q_2 = \eta, \qquad \dots (11)$$

$$q_3 = \theta.$$

If x, y denote the cartesian co-ordinates of any other point B of the rod, distant r from the previous fixed point A, we have the connection between x, y and ξ, η, θ as

$$x = \xi + r\cos\theta \qquad y = \eta + r\sin\theta \qquad \dots (12)$$

$$= q_1 + r\cos q_3 \qquad = q_2 + r\sin q_3$$

relative to axes OXY. In the transformation relations, r is simply a number. It cannot be treated as distinct co-ordinate; for, r cannot be varied without violating the constraint that the distance of two particles must remain constant in time. That is why we suppress explicit reference to such numbers.

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In general, we can always express generalised co-ordinates as some functions of cartesian co-ordinates, and possibly function of time (*cf.* eqs. 8, 9, 10, 11, 12):

$$q_1 = q_1 (x_1, y_1, z_1 ; x_2, y_2, z_2 ; \ldots z_N, t)$$
$$q_2 = q_2 (x_1, y_1, z_1 ; \ldots \ldots \ldots \ldots z_N, t)$$
$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$
$$q_{3N} = q_{3N} (x_1, y_1, z_1 ; \ldots\ldots\ldots\ldots z_N, t). \qquad \ldots (13)$$

for a system of N particles free from constraints which require the specification of $3N$ generalised co-ordinates. Eqs. (13) are then the *transformation equations* from a set of $3N$ cartesian co-ordinates to $3N$ generalised co-ordinates. If constraints are present, the number of generalised co-ordinates will be reduced accordingly. And, since we accept the co-ordinates $q_1, q_2, \ldots q_{3N}$ as specifying the configuration of the system, it must be possible to express cartesian co-ordinates in terms of them and *vice-versa* :

$$x_1 = x_1 (q_1, q_2, \ldots\ldots q_{3N}, t)$$
$$y_1 = y_1 (q_1, q_2, \ldots\ldots q_{3N}, t) \qquad \ldots (14)$$
$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$
$$z_N = z_N (q_1, q_2, \ldots\ldots q_{3N}, t).$$

The necessary and sufficient condition that the transformation from a set of co-ordinates q_j ($j = 1, 2, \ldots 3N$) to the set $(x_1, y_1, \ldots z_N)$ is effective is that the *Jacobian* determinant J of eqs. (13) be different from zero at all points, *i.e.*,

$$J\left(\frac{q_1, q_2, \ldots q_{3N}}{x_1, y_1, \ldots z_N}\right) \equiv \left(\frac{\partial (q_1, q_2, \ldots q_{3N})}{\partial (x_1, y_1, \ldots z_N)}\right) = \begin{vmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_2}{\partial x_1} & \cdots & \frac{\partial q_{3N}}{\partial x_1} \\ \frac{\partial q_1}{\partial x_2} & \frac{\partial q_2}{\partial x_2} & \cdots & \frac{\partial q_{3N}}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial q_1}{\partial z_N} & \frac{\partial q_2}{\partial z_N} & \cdots & \frac{\partial q_{3N}}{\partial z_N} \end{vmatrix} \neq 0$$

If this condition fails to hold, eqs. (13) do not define a consistent set of generalised co-ordinates. If the Jacobian determinant does not vanish, the co-ordinates q_j are as effective as the cartesian co-ordinates in describing the kinematical motion of the system and are most convenient to use.

Some examples of mechanical systems and the generalised co-ordinates to be associated are given below :

System	Generalised Co-ordinates
1. Simple pendulum	θ ; the angle which the pendulum makes with vertical line through point of suspension.
2. Fly-wheel	θ ; the angle between a definite radius of the fly wheel and fixed line perpendicular to the axis.
3. Particles on the surface of a sphere.	θ, φ ; the usual polar angle of a point on the sphere.
4. Beads of an abacus	x ; the cartesian co-ordinate along the horizontal wire.
5. Hydrogen molecule	x, y, z, φ, ψ ; x, y, z are the cartesian co-ordinates of the centre of molecule ; φ and ψ the angles of rotation about two mutually perpendicular axes through centre.
6. Particles moving on inside surface of a cone.	r, θ ; r the radius vector drawn from the vertex as origin to the position of the particle and angle θ of the radius vector with a fixed slant edge of the cone.

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