Constraints :

A constrained motion is a motion which cannot proceed arbitrarily in any manner. Particle motion is restricted to occur only, for example, along some specified path, or on a surface (plane or curved) arbitrarily oriented in space. Motion along a specified path is the simplest example of a constrained motion. Here, one co-ordinate is sufficient to describe the motion in contrast to the situation where the particle is free to move in space and then three co-ordinates are needed to describe its motion. *Thus imposing constraints on a mechanical system is to simplifiy the mathematical description.* When a particle (bead) is made to slide on a wire, constraints require that the position of the bead lie on the wire. Condition imposed on the system by the constraints can, in most cases, be written down mathematically as a relation satisfied by the co-ordinates of the particle at any time. This is the way in which constraints reduce the number of co-ordinates needed to specify the configuration of a system.

☆ **Example 1 :** Let us consider the motion of a simple pendulum confined to move in the vertical plane. We would need only two co-ordinates (cartesian co-ordinates x and y or polar co-ordinates r and θ with respect to the point of suspension, O, as origin) to locate the position of the bob in motion. However, motion of the bob is not free but takes place under a constraint that the distance l of the bob is to remain the same from O all the time. This condition imposed by the constraint can be expressed in the form of an equation either between x and y or r and θ :

$$x^2 + y^2 = l^2,$$

or

$$r = l. \qquad \dots (1)$$

In plane polar co-ordinates, the equation looks simpler. Again one co-ordinate θ, in polar co-ordinates, would suffice to describe the motion. Note that we have utilised eq. (1) to reduce the number of co-ordinates which, otherwise, would have been two.

☆ **Example 2 :** Let us take up another example. A particle moving in space requires three co-ordinates to determine its position at any instant. If we restrict its movement on the surface of a sphere, there exists a relation between these co-ordinates. Again we shall see that spherical polar co-ordinates can be used here with advantage :

$$x^2 + y^2 + z^2 = a^2,$$

or

$$r = a \qquad \dots (2)$$

in which a is the radius of the sphere. Each of eqs. (2) is the equation of the surface of the sphere with its centre at origin. We may use the equation of constraints to eliminate one co-ordinate from the set of three co-ordinates and we are left with two co-ordinates θ and φ to describe the position of the particle completely. This can be performed conveniently by using spherical polar co-ordinates when we select θ and φ as independent co-ordinates.

☆ **Example 3 :** We take up a third example of double pendulum shown in fig. (1.5) moves in a vertical plane (which is a two-particle system connected by an inextensible light rod and suspended by a similar rod fastened to one of the particles). We would require four co-ordinates (two for each particle) to describe the system completely : but two of them are eliminated by the equations of constraints *viz*. distances to particle 1 should be constant when measured from point of suspension O and from particle 2. Then two convenient co-ordinates would be the angles θ_1 and θ_2 shown in the figure.

Fig. 1.5.

☆ **Example 4 :** As a last and final example, we consider a rigid body. A rigid body is defined as a system of particles in which the relative distances of the constituent particles are fixed and cannot vary with time. In this case, constraints are expressed by equation of the form

$$r_{ij} = c_{ij},$$

in which c_{ij} are constants and r_{ij} denote the distance between i^{th} and j^{th} particles. In terms of co-ordinates $\mathbf{r}_i$ $(x_i\, y_i\, z_i)$ and $\mathbf{r}_j$ $(x_j\, y_j\, z_j)$ with respect to the origin we have these conditions expressed as :

$$(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = (c_{ij})^2. \quad \text{... (3}$$

All these instances serve to demonstrate that each constraint, which can be expressed in the form of an equation like (1), (2), (3) or in the general case of an N particle system as an equation connecting the co-ordinates of the particles having the form

$$f(x_1, y_1, z_1\,;\; x_2, y_2, z_2 : x_n, y_n, z_n\,;\; t) = 0 \quad \text{... (4}$$

where time t may occur in case of constraints which may vary with time, and enables us to eliminate one of the co-ordinates by choosing co-ordinate in such a manner that it is held constant by the constraint. For a rigid body containing N particles, there are $\frac{1}{2} N (N-1)$ pairs of particles. It is not difficult to show that it is sufficient to specify the mutual distances of the $(3N-6)$ pairs, if $N > 3$. Hence we can replace the $3N$ cartesian co-ordinates originally needed, had the system been free from constraints, by 3 co-ordinates of centre of mass and 3 co-ordinates describing the orientation of the body (refer to chapter on rigid body motion). Now $(3N-6)$ distances are constant and the problem is reduced to one of finding the motion in terms of only 3 plus 3 = 6 co-ordinates.

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