4.5 GROUP VELOCITY

A group consists of a number of waves of different frequencies superimposed upon each other. For example, white light consists of a continuous visible wavelength spectrum ranging from about 3000 Å in the violet to about 7000 Å in the red region. *The velocity with which the slowly varying envelop of the modulated pattern due to a group of waves travels in a medium is known as the group velocity.* The importance of the group velocity lies in the fact that this is the velocity with which the energy in the wave group is transmitted.

To derive an expression for group velocity consider a group of waves consisting of only two components of equal amplitude and having angular frequencies ω_1 and ω_2 differing by a small amount and represented by the equations

$$y_1 = a \cos (\omega_1 t - k_1 x)$$

and

$$y_2 = a \cos (\omega_2 t - k_2 x)$$

where $\frac{\omega_1}{k_1}$ and $\frac{\omega_2}{k_2}$ represent their respective phase (wave) velocities. ω being equal to $2\pi\nu$ and $k = \frac{2\pi}{\lambda}$. It is further supposed that $\frac{\omega_1}{k_1} \neq \frac{\omega_2}{k_2}$ *i.e.* it is a *dispersive medium*. The resultant amplitude is given by

$$y = y_1 + y_2 = a [\cos (\omega_1 t - k_1 x) + \cos (\omega_2 t - k_2 x)]$$

$$= 2a \cos \left[\frac{(\omega_1 + \omega_2)}{2} t - \frac{(k_1 + k_2)}{2} x \right] \cos \left[\frac{(\omega_1 - \omega_2)}{2} t - \frac{(k_1 - k_2)}{2} x \right]$$

$$= 2a \cos (\omega t - kx) \cos \left(\frac{\Delta \omega t}{2} - \frac{\Delta k x}{2} \right)$$

where $\omega = \frac{\omega_1 + \omega_2}{2}$, $k = \frac{k_1 + k_2}{2}$

$$\Delta\omega = \omega_1 - \omega_2 \text{ and } \Delta k = k_1 - k_2$$

The resultant wave has two parts.

(*i*) **Phase velocity.** A wave of frequency ω, propagation constant k and velocity

$$v_p = \frac{\omega}{k} = \frac{2\pi\nu}{2\pi/\lambda} = \nu\lambda \qquad ...(i)$$

This is the *phase velocity* or wave velocity.

(*ii*) **Group velocity.** A second wave of frequency $\frac{\Delta\omega}{2}$, propagation constant $\frac{\Delta k}{2}$ and velocity

$$v_g = \frac{\Delta\omega}{\Delta k}$$

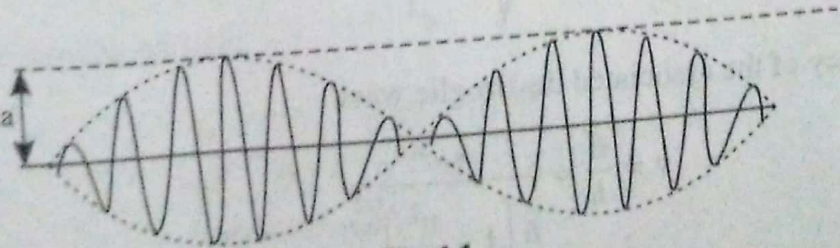

Fig. 4.1

It consists of a group of waves of the first type and is a very slowly moving envelope of frequency $\frac{\Delta\omega}{2}$ and propagation constant $\frac{\Delta k}{2}$. The modulated pattern moves with a velocity

$$v_g = \frac{\Delta\omega}{\Delta k} \quad \text{known as the } \textit{group velocity}.$$

If $\Delta\omega$ and Δk are very small, we can put

Group velocity $$v_g = \frac{\partial\omega}{\partial k}$$

Also $$v_g = \frac{2\pi\,\partial\nu}{2\pi\partial\left(\frac{1}{\lambda}\right)} = -\lambda^2\frac{\partial\nu}{\partial\lambda}.$$

Relation between phase velocity and group velocity. We have seen that

$$v_p = \frac{\omega}{k} \qquad \therefore\ \omega = kv_p$$

and $$v_g = \frac{d\omega}{dk} = \frac{dkv_p}{dk} = v_p + \frac{k\,dv_p}{dk}$$

But $$k = \frac{2\pi}{\lambda} \qquad \therefore\ dk = -\frac{2\pi}{\lambda^2}\,d\lambda$$

and $$\frac{k}{dk} = -\frac{\lambda}{d\lambda}$$

Hence $$v_g = v_p - \lambda\frac{dv_p}{d\lambda}$$

Particle velocity. *It is the velocity with which the particle travels.*

Relation between group velocity and particle velocity. A particle moving with a velocity is supposed to consist of a group of waves according to de-Broglie hypothesis. For a material particle of rest mass m_0 moving with a velocity v and having an effective mass m the total energy E is given

$$E = mc^2 = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

and momentum $$p = mv = \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}$$

The frequency of the associated de-Broglie wave

$$\nu = \frac{E}{h} = \frac{m_0c^2}{h\left(1-\frac{v^2}{c^2}\right)^{1/2}}$$

Date — 14-01-2022

EC-(23)

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2. Subject — PHYSICS
3. Paper — V
4. TDC — III
5. Topic — Phase & group velocities and its relationships
6. Date — 14-01-2022