6.4 TIME-DEPENDENT SCHRÖDINGER'S WAVE EQUATION

A plane wave moving in the X-direction is represented in the exponential form as

$$y = Ae^{i(\omega t - kx)} \quad ...(i)$$

where $\omega = 2\pi\nu$ (ν = frequency of the wave) and k the propagation constant $= \frac{2\pi}{\lambda}$ (λ = wavelength of the wave).

According to quantum theory, the energy of a wave

$$E = h\nu$$

and therefore

$$\nu = \frac{E}{h}$$

and

$$\omega t = 2\pi\nu t = 2\pi \frac{E}{h} t = \frac{Et}{h/2\pi} = \frac{Et}{\hbar}$$

and momentum

$$p_x = \frac{h}{\lambda} \text{ or } \frac{1}{\lambda} = \frac{p_x}{h}$$

$$\therefore \quad kx = \frac{2\pi}{\lambda} x = 2\pi \frac{p_x x}{h} = \frac{p_x}{h/2\pi} x = \frac{p_x}{\hbar} x$$

where

$$\hbar = \frac{h}{2\pi} = 1.054 \times 10^{-34} \text{ Js}.$$

Substituting in (i) we can express a plane wave moving in the X-direction as

$$y = Ae^{i\left(\frac{E}{\hbar}t - \frac{p_x x}{\hbar}\right)}$$

$$= Ae^{\frac{i}{\hbar}(Et - xp_x)} \quad ...(ii)$$

By analogy between photons and electron beams, we assume that the electrons in a beam travelling in the X-direction with constant momentum p_x and energy E are controlled by *pilot* waves with a wave function Ψ which plays the same role for electrons and other particles as the displacement y for sound waves or electric field vector E for electro-magnetic waves.

Thus $\Psi(x, t)$ as a function of x and t denotes the amplitude of matter wave at a point x at a time t and is given by an equation similar to equation *(ii)* *i.e.*,

$$\Psi(x,t) = Ae^{\frac{i}{\hbar}(Et - xp_x)}$$

However, in wave mechanics, we use the exponential form with a *negative sign* *i.e.*,

$$\psi(x,t) = Ae^{\frac{-i}{\hbar}(Et - xp_x)}$$

$$= Ae^{\frac{i}{\hbar}(xp_x - Et)} \quad \text{...(iii)}$$

for a wave moving along the positive direction of X-axis. We represent a wave moving along the negative direction of X-axis by the equation

$$\Psi(x,t) = Ae^{\frac{-i}{\hbar}(xp_x - Et)}$$

This merely introduces a phase difference of π from the notation used for sound waves, electromagnetic and other waves.

Differentiating equation *(iii)* with respect to time and writing only Ψ for $\Psi(x, t)$, we have

$$\frac{\partial \Psi}{\partial t} = -\frac{i}{\hbar} EAe^{\frac{i}{\hbar}(xp_x - Et)} = -\frac{i}{\hbar} E\Psi \quad \text{...(iv)}$$

or

$$E\Psi = -\frac{\hbar}{i}\frac{\partial \Psi}{\partial t} \quad \text{...(v)}$$

Differentiating equation *(iii)* with respect to x, we have

$$\frac{\partial \Psi}{\partial x} = \frac{i}{\hbar} p_x Ae^{\frac{i}{\hbar}(xp_x - Et)}$$

and

$$\frac{\partial^2 \Psi}{\partial x^2} = -\frac{p_x^2}{\hbar^2} Ae^{\frac{i}{\hbar}(xp_x - Et)} = -\frac{p_x^2}{\hbar^2}\Psi$$

or

$$p_x^2 \Psi = -\hbar^2 \frac{\partial^2 \Psi}{\partial x^2} \quad \text{...(vi)}$$

C-(29)

1. Name — Dr. [signature]
2. Subject — PHYSICS
3. Class — TDC-III
4. Paper — V
5. Topic — Time dependent Schroedinger wave equation
6. Date — 19-01-22